It's our last Fun Friday! (sniff)

Application of Stokes' Thm to Physics:

Maxwell's Equations

① Gauss's Law for Electricity If Ω is a region in $\mathbb{R}^3$,

$$\underset{\partial\Omega}{\iint} \underset{\text{Electric Field}}{E} \cdot \hat{n}\, dA = \underset{\text{vacuum permittivity}}{\frac{1}{\varepsilon_0}} \underset{\Omega}{\iiint} \underset{\text{charge density}}{\rho}\, dV$$

$$= \underset{\Omega}{\iiint} (\operatorname{div} E)\, dV = \frac{1}{\varepsilon_0} \underset{\Omega}{\iiint} \rho\, dV$$

Since this is true for any region Ω,

this means $\boxed{\operatorname{div} E = \frac{1}{\varepsilon_0}\rho}$ at every point

derivative version of Gauss' Law

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② Gauss' Law of Magnetism.

If Ω is a region in $\mathbb{R}^3$,

$$\underset{\partial\Omega}{\iint} \underset{\text{magnetic field}}{B} \cdot n\, dA = 0 \iff \underset{\Omega}{\iiint} \operatorname{div} B\, dV = 0$$

$$\iff \boxed{\operatorname{div} B = \nabla \cdot B = 0 \text{ at every pt.}}$$

derivative version of Gauss's Law of Magnetism.

③ Faraday's Law of Induction

If Σ_1' is a surface in $\mathbb{R}^3$,

$$\underbrace{\oint_{\partial\Sigma_1} E\cdot ds}_{\text{Stokes}} = -\frac{d}{dt}\iint_{\Sigma_1'} B\cdot n\,dA$$

$$\iint_{\Sigma_1'} (\nabla\times E)\cdot n\,dA = -\frac{d}{dt}\iint_{\Sigma_1'} B\cdot n\,dA$$

Since this true for any surface

$$\Rightarrow \quad \boxed{\nabla\times E = -\frac{d}{dt}B}$$

derivative version of Faraday!

④ Ampere's Law: for any surface Σ_1'

$$\oint_{\partial\Sigma_1'} B\cdot ds = \mu_0 \underbrace{\iint_{\Sigma_1'} J\cdot n\,dA}_{\text{total current through } \Sigma_1'} + \frac{1}{c^2}\frac{\partial}{\partial t}\iint_{\Sigma_1'} E\cdot n\,dA$$

B: magnetic field

μ_0: vacuum permeability

J: current density

c: speed of light, $\varepsilon_0\mu_0 = \frac{1}{c^2}$

$$\iint_{\Sigma_1'} (\nabla\times B)\cdot n\,dA = \text{''}$$

Since this is true for all surfaces Σ_1',

$$\Rightarrow \quad \boxed{\nabla\times B = \mu_0 J + \frac{1}{c^2}\frac{\partial E}{\partial t}}$$

derivative version of Ampere's Law.

Maxwell Eqns:

$$\nabla \cdot E = \frac{\rho}{\varepsilon_0}$$

$$\nabla \cdot B = 0$$

$$\nabla \times E = -\frac{\partial B}{\partial t}$$

$$\nabla \times B = \mu_0 J + \frac{1}{c^2}\frac{\partial E}{\partial t}$$

Form version:

$$E = E_1\, dx + E_2\, dy + E_3\, dz$$

$$B = B_1\, dy \wedge dz + B_2\, dz \wedge dx + B_3\, dy \wedge dz$$

$\vdots$

Let $F = B + E \wedge dt$

$$\boxed{dF = 0}$$ First two equations

$$J = \rho\, dt + \cdots$$

$$\boxed{* d * F = J}$$

conversion between 1-form $\Leftrightarrow$ 2-form